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Two problems with mixed boundary conditions for an elastic orthotropic strip☆

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Abstract

Two problems are considered for an elastic orthotropic strip: the contact problem and the crack problem. Both problems are reduced to integral equations of the first kind with different kernels, containing a singularity: logarithmic for the first problem and singular for the second problem. Regular and singular asymptotic methods are employed to construct approximate solutions of these integral equations. Numerical results are presented.

Similar problems were considered earlier for an elastic isotropic strip (the contact problem in Ref. 1 and the crack problem in Ref. 2).

1. Formulation of the contact problem

Suppose an elastic orthotropic strip of thickness h occupies the region $|x| < \infty$, $0 \leq y \leq h$. The lower boundary of the strip $y = 0$ is hinged and a rigid punch is indented into the upper boundary $y = h$ by a force P. The area of contact of the punch with the strip boundary is defined by the inequality $|x| \leq a$. The axis of action of the force is parallel to the y axis and is situated a distance $e < a$ to the right of it. We will neglect the friction forces in the contact area.

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[1] From the Editorial Board. January 1, 2006 was the seventieth birthday of Viktor Mikhailovich Aleksandrov, chief scientist at the Institute of Problems in Mechanics of the Russian Academy of Sciences, Doctor of Physical-Mathematical Sciences, Professor, honoured scientist of the Russian Federation, State prize laureate of the Russian Federation, and Acting Member of the Russian Academy of Natural Sciences (Physics Section). The main areas of his scientific activities are: the mechanics of contact interactions and the mechanics of the fracture of deformable solids, contact problems of tribology, the theory of stress concentration in the region of fine defects, problems of the mechanics of continua with mixed boundary conditions, and integral, integrodifferential and functional equations of problems of mathematical physics. The list of his publications contains more than 350 titles, among which there are nine monographs (his monographs are listed in the Further reading). He supervised 30 Candidate dissertations. Seven of his students went on to complete Doctorate dissertations.

He is a member of the National Committee of Theoretical and Applied Mechanics, the National Committee on Tribology, the Scientific Committee of the Russian Academy of Sciences on the mechanics of deformable solids and of the Interdepartmental Scientific Committee on Tribology. He was awarded the Order of Honour.

The editorial board and editorial staff of the *Journal of Applied Mathematics and Mechanics*, his colleagues and students warmly congratulate him on his Jubilee, and wish him health and new achievements in science.

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By virtue of the above, we will have the following boundary conditions for the contact problem

$$\begin{aligned} &\upsilon(x,0) = \tau_{xy}(x,0) = 0, \quad \tau_{xy}(x,h) = 0; \quad |x| < \infty \\ &\sigma_y(x,h) = 0, \quad |x| > a; \quad \upsilon(x,h) = -[\delta + \alpha x - f(x)], \quad |x| \le a \end{aligned} \tag{1.1}$$

Here υ is the displacement along the y axis, τ_{xy} and σ_y are the shear and normal stresses, δ is the translation of displacement of the punch along the negative direction of the y axis due to the action of the force P, α is the angle of rotation of the punch due to the action of the moment Pe, and $f(x)$ is a function describing the shape of the base of the punch.

We will assume that the strip is under plane-deformation conditions. The formulae of Hooke's law for the strip can then be represented in the form[3]

$$\begin{aligned} &\varepsilon_x = \frac{1-\nu_{31}\nu_{13}}{E_1}\sigma_x - \frac{\nu_{21}+\nu_{31}\nu_{23}}{E_2}\sigma_y, \quad \varepsilon_y = -\frac{\nu_{12}+\nu_{32}\nu_{13}}{E_1}\sigma_x + \frac{1-\nu_{32}\nu_{23}}{E_2}\sigma_y \\ &\gamma_{xy} = \frac{\tau_{xy}}{G_{12}}; \quad \frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2}, \quad \frac{\nu_{32}\nu_{13}}{E_1} = \frac{\nu_{31}\nu_{23}}{E_2} \end{aligned} \tag{1.2}$$

Here σ_x is the normal stress, ε_x, ε_y and γ_{xy} are the deformations, ν_{ij} is Poisson's ratio, E_1 and E_2 are Young's moduli and G_{12} is the shear modulus (seven independent mechanical quantities in all). We must add to formulae (1.2) the equilibrium equations

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} = 0, \quad \frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} = 0 \tag{1.3}$$

the deformation compatibility equation

$$\frac{\partial^2\varepsilon_x}{\partial y^2} + \frac{\partial^2\varepsilon_y}{\partial x^2} = 2\frac{\partial^2\gamma_{xy}}{\partial x\partial y} \tag{1.4}$$

and also the relation

$$\upsilon = \int\varepsilon_y dy \tag{1.5}$$

2. The integral equation of the contact problem

$$\sigma_x = \frac{\partial^2\Phi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2\Phi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2\Phi}{\partial x\partial y} \tag{2.1}$$

Assuming we will satisfy the equilibrium equations (1.3). Then, substituting expressions (2.1) into expressions (1.2), and then substituting the converted expressions (1.2) into Eq. (1.4), we obtain the following differential equation for the Airy function

$$\begin{aligned} &\frac{\partial^4\Phi}{\partial y^4} + 2A\frac{\partial^4\Phi}{\partial x^2\partial y^2} + B\frac{\partial^4\Phi}{\partial x^4} = 0 \\ &A = \frac{E_1[E_2 - G_{12}(\nu_{21}+\nu_{31}\nu_{23})]}{G_{12}E_2(1-\nu_{31}\nu_{13})}, \quad B = \frac{E_1(1-\nu_{32}\nu_{23})}{E_2(1-\nu_{31}\nu_{13})} \end{aligned} \tag{2.2}$$

We will further assume that $A > 0$ and $B > 0$.

We will first consider an auxiliary problem with the boundary conditions

$$\begin{aligned} &\upsilon(x,0) = \tau_{xy}(x,0) = 0, \quad \tau_{xy}(x,h) = 0, \quad \sigma_y(x,h) = -\tilde{q}(x); \quad |x| < \infty \\ &\tilde{q}(x) = q(x), \quad |x| \le a; \quad \tilde{q}(x) = 0, \quad |x| > a \end{aligned} \tag{2.3}$$

Investigating the Airy function in the form of a Fourier integral in x

$$\Phi(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Phi^*(y,\alpha)e^{-i\alpha x}d\alpha \tag{2.4}$$

and assuming for the present that $A^2 > B$, from Eq. (2.2) we obtain the following expression for the Fourier transform

$$\begin{gathered}\Phi^*(y,\alpha) = c_1\operatorname{sh}(\kappa_1\alpha y) + c_2\operatorname{ch}(\kappa_1\alpha y) + c_3\operatorname{sh}(\kappa_2\alpha y) + c_4\operatorname{ch}(\kappa_2\alpha y)\\ \kappa_{1,2} = \sqrt{A \pm \sqrt{A^2 - B}}\end{gathered} \tag{2.5}$$

where $c_1, \ldots, c_4$ are functions of α, which must be determined from the fact that boundary conditions (2.3) are satisfied. Carrying out this determination, we obtain, as a result of solving auxiliary problem (2.3),

$$\begin{gathered}\upsilon(x,h) = -\frac{1}{2\pi\theta}\int_{-\infty}^{\infty}Q(\alpha)\frac{L(\alpha h)}{\alpha}e^{-i\alpha x}d\alpha, \quad Q(\alpha) = \int_{-a}^{a}q(\xi)e^{i\alpha\xi}d\xi\\ L(u) = \frac{\kappa_2 - \kappa_1}{\kappa_2\operatorname{cth}(\kappa_1 u) - \kappa_1\operatorname{cth}(\kappa_2 u)}, \quad \theta = \frac{E_2\kappa_1\kappa_2}{(\kappa_1 + \kappa_2)(1 - \nu_{32}\nu_{23})} \quad (A^2 > B)\end{gathered} \tag{2.6}$$

where $Q(\alpha)$ is the Fourier transform of the discontinuous function $\tilde{q}(x)$ of the form (2.3).

We now return to the main problem. Satisfying the last boundary condition of (1.1) using relations (2.6) (the remaining conditions of (1.1) have already been satisfied in the course of obtaining the first relation of (2.6)), we arrive at the following integral equation in the contact-pressure function $q(x)$

$$\int_{-a}^{a}q(\xi)K\left(\frac{\xi - x}{h}\right)d\xi = \pi\theta[\delta + \alpha x - f(x)], \quad |x| \le a; \quad K(t) = \int_0^{\infty}\frac{L(u)}{u}\cos ut\,du \tag{2.7}$$

We once again present the expressions for the function $L(u)$ and the quantity θ for $A^2 = B$ and $A^2 < B$

$$\begin{gathered}L(u) = \frac{\operatorname{ch}(2Au) - 1}{\operatorname{sh}(2Au) + 2Au}, \quad \theta = \frac{E_2 A}{2(1 - \nu_{32}\nu_{23})} \quad (A^2 = B)\\ L(u) = \frac{d[\operatorname{ch}(2cu) - \cos(2du)]}{d\operatorname{sh}(2cu) + c\sin(2du)}, \quad \theta = \frac{E_2(c^2 + d^2)}{2c(1 - \nu_{32}\nu_{23})} \quad (A^2 < B)\\ c = \frac{1}{\sqrt{2}}b, \quad d = \frac{D}{\sqrt{2}b}, \quad D = \sqrt{B - A^2}, \quad b = \sqrt{A + \sqrt{A^2 + D^2}}\end{gathered} \tag{2.8}$$

Note that integral Eq. (2.7) does not change its form for the cases when $A^2 = B$ and $A^2 < B$.

3. The asymptotic solution for a considerable relative thickness of the strip

For all three cases of the relation between the quantities A^2 and B we have the formulae

$$\begin{gathered}L(u) = 1 + O(e^{-2\kappa u}), \quad u \to \infty\\ \kappa = \inf(\kappa_1, \kappa_2)\ (A^2 > B), \quad \kappa = A\ (A^2 = B), \quad \kappa = c\ (A^2 < B)\\ L(u) = lu + O(u^5), \quad u \to 0\\ l = \frac{\kappa_1\kappa_2}{\kappa_1 + \kappa_2}\ (A^2 > B), \quad l = \frac{A}{2}\ (A^2 = B), \quad l = \frac{c^2 + d^2}{2c}\ (A^2 < B)\end{gathered} \tag{3.1}$$

On the basis of relations (3.1), the kernel of integral Eq. (2.7) can be represented in the form[1]

$$K(t) = -\ln t - \sum_{i=0}^{\infty} d_i t^{2i}$$
$$d_0 = \int_0^{\infty} \frac{1 - L(u) - e^{-u}}{u} du, \quad d_i = \frac{(-1)^i}{(2i)!} \int_0^{\infty} [1 - L(u)] u^{2i-1} du, \quad i = 1, 2, \ldots \tag{3.2}$$

The series in expression (3.2) converges absolutely when $|t| < 2\kappa$.

Changing in integral Eq. (2.7) to dimensionless variables and notation using the formulae

$$\xi' = \frac{\xi}{a}, \quad x' = \frac{x}{a}, \quad \varphi(x') = \frac{q(x)}{\theta}, \quad g(x') = \frac{\delta + \alpha x - f(x)}{a}, \quad \lambda = \frac{h}{a} \tag{3.3}$$

and substituting expression (3.2) for $K(t)$ into it, we will have

$$-\int_{-1}^{1} \varphi(\xi) \ln\left|\frac{\xi - x}{\lambda}\right| d\xi = \pi g(x) + \sum_{i=0}^{\infty} \frac{d_i}{\lambda^{2i}} \int_{-1}^{1} \varphi(\xi)(\xi - x)^{2i} d\xi, \quad |x| \le 1 \tag{3.4}$$

Here and henceforth the primes on dimensionless variables will be omitted.

We will seek a solution of integral Eq. (3.4) in the form of a series

$$\varphi(x) = \sum_{n=0}^{\infty} \varphi_n(x) \lambda^{-2n} \tag{3.5}$$

Substituting expression (3.5) into integral Eq. (3.4) and equating terms of like powers of λ^{-2} on the left and right, we arrive[1] at an infinite system of successively solvable integral equations in the functions $\varphi_n(x)$. We present the result of this solution

$$\varphi(x) = \frac{N_0}{\pi\sqrt{1-x^2}}\left[1 - \frac{2d_1}{\lambda^2}\left(\frac{1}{2} - x^2\right) - \frac{4d_2}{\lambda^4}\left(\frac{7}{8} - x^2 - x^4\right) - \right.$$
$$\left. - \frac{3d_1 d_2}{\lambda^6}\left(\frac{1}{2} - x^2\right) - \frac{6d_3}{\lambda^6}\left(\frac{13}{8} + \frac{3}{4}x^2 - \frac{9}{2}x^4 - x^6\right)\right] -$$
$$- \frac{1}{\pi\sqrt{1-x^2}} \int_{-1}^{1} \sqrt{1-t^2}\, g'(t)\left(\frac{1}{t-x} + \frac{2d_1 x}{\lambda^2} + \frac{2}{\lambda^4}[-d_1^2 x + d_2(3t - 2x + \right.$$
$$+ 2xt^2 - 6x^2 t + 6x^3)] + \frac{1}{\lambda^6}\left\{2x[d_1^3 - d_1 d_2(1 + 2t^2 + 6x^2)] + \right.$$
$$+ 3d_3\left[5t + 5t^3 - \frac{1}{2}(11 + 18t^2 - 4t^4)x + (5t - 10t^3)x^2 + \right.$$
$$\left.\left.\left. + (5 + 20t^2)x^3 - 20x^4 t + 10x^5\right]\right\}\right) dt + O\left(\frac{1}{\lambda^8}\right) \tag{3.6}$$

The value of the dimensionless indenting force

$$N_0 = \frac{P}{\theta a} = \frac{1}{\theta a}\int_{-a}^{a} q(\xi) d\xi = \int_{-1}^{1} \varphi(\xi) d\xi \tag{3.7}$$

must be determined from the relation[1]

$$N_0 \ln 2\lambda = \int_{-1}^{1} \frac{g(t)dt}{\sqrt{1-t^2}} + \frac{1}{\pi}\sum_{i=0}^{\infty} \frac{d_i}{\lambda^{2i}} \int_{-1}^{1} \frac{dt}{\sqrt{1-t^2}} \int_{-1}^{1} \varphi(\xi)(\xi - t)^{2i} d\xi \tag{3.8}$$

Substituting expression (3.6) into (3.8), we obtain

$$N_0 = \left[\ln 2\lambda - d_0 - \frac{d_1}{\lambda^2} - \frac{d_1^2}{4\lambda^4} - \frac{9d_2}{4\lambda^4} - \frac{2d_1 d_2}{\lambda^6} - \frac{25 d_3}{4\lambda^6} + O\left(\frac{1}{\lambda^8}\right)\right]^{-1} \times$$

$$\times\left\{\int_{-1}^{1} \frac{g(t)dt}{\sqrt{1-t^2}} + \int_{-1}^{1} \sqrt{1-t^2} g'(t) t \left[\frac{d_1}{\lambda^2} + \frac{d_2}{\lambda^4}\left(\frac{7}{2} + t^2\right) + \right.\right. \tag{3.9}$$

$$\left.\left. + \frac{d_3}{\lambda^6}\left(\frac{39}{4} + 8t^2 + t^4\right)\right] dt + O\left(\frac{1}{\lambda^8}\right)\right\}$$

We again obtain an expression for the moment Pe acting on the punch. We introduce a dimensionless moment by the relation

$$N_1 = \frac{Pe}{\theta a^2} = \frac{1}{\theta a^2} \int_{-a}^{a} \xi q(\xi) d\xi = \int_{-1}^{1} \xi \varphi(\xi) d\xi \tag{3.10}$$

Substituting expression (3.6) into (3.10), we obtain

$$N_1 = \int_{-1}^{1} \sqrt{1-t^2} g'(t) \left[1 - \frac{d_1}{\lambda^2} + \frac{d_1^2}{\lambda^4} - \frac{d_2}{\lambda^4}\left(\frac{5}{2} + 2t^2\right) - \frac{d_1^3}{\lambda^6} + \right.$$

$$\left. + \frac{d_1 d_2}{\lambda^6}\left(\frac{11}{2} + 2t^2\right) - \frac{3d_3}{\lambda^6}\left(\frac{9}{4} + 3t^2 + t^4\right)\right] dt + O\left(\frac{1}{\lambda^8}\right) \tag{3.11}$$

4. The asymptotic solution for a small relative thickness of the strip

We will confine ourselves to constructing the principal term of the asymptotic form of the solution of integral Eq. (2.7). As was shown in Ref. 1, it is given by the expression

$$\varphi(x) = \varphi^{(1)}\left(\frac{1+x}{\lambda}\right) + \varphi^{(2)}\left(\frac{1-x}{\lambda}\right) - \varphi^{(0)}\left(\frac{x}{\lambda}\right) \tag{4.1}$$

where the functions $\varphi^{(1)}(t)$, $\varphi^{(2)}(t)$ and $\varphi^{(0)}(t)$ must be found from the integral equations

$$\int_{0}^{\infty} \varphi^{(k)}(\tau) K(\tau - t) d\tau = \frac{\pi}{\lambda} g[(-1)^{k+1}(\lambda t - 1)], \quad 0 \le t < \infty; \quad k = 1, 2$$

$$\int_{-\infty}^{\infty} \varphi^{(0)}(\tau) K(\tau - t) d\tau = \frac{\pi}{\lambda} g(\lambda t), \quad -\infty < t < \infty \tag{4.2}$$

In formula (4.1) and here as before we have taken the dimensionless variables and the notation of (3.3).

The solution of the first two integral equations (4.2) can be found by the Wiener–Hopf method[4]. The solution of the third integral equation of (4.2) can be found using the convolution theorem for a Fourier integral transformation[5].

Note that, in representation (4.1), solutions of the boundary-layer type $\varphi^{(1)}(t)$ and $\varphi^{(2)}(t)$ automatically match with the penetrating solution $\varphi^{(0)}(t)$.

5. Numerical example

We will confine ourselves to considering the case of a plane inclined punch, i.e. the case when $f(x) \equiv 0$. In this case

$$g(x) = \varepsilon + \alpha x, \quad \varepsilon = \delta/a \tag{5.1}$$

Substituting expression (5.1) into formulae (3.9) and (3.11) we obtain

$$\begin{aligned} N_0 &= \pi\varepsilon\left[\ln 2\lambda - d_0 - \frac{d_1}{\lambda^2} - \frac{d_1^2}{4\lambda^4} - \frac{9d_2}{4\lambda^4} - \frac{2d_1d_2}{\lambda^6} - \frac{25d_3}{4\lambda^6} + O\left(\frac{1}{\lambda^8}\right)\right]^{-1} \\ N_1 &= \frac{\pi\alpha}{2}\left[1 - \frac{d_1}{\lambda^2} + \frac{d_1^2}{\lambda^4} - \frac{3d_2}{\lambda^4} - \frac{d_1^3}{\lambda^6} + \frac{6d_1d_2}{\lambda^6} - \frac{75d_3}{8\lambda^6} + O\left(\frac{1}{\lambda^8}\right)\right] \end{aligned} \tag{5.2}$$

The solution of the third integral equation of (4.2) for the case (5.1) can be found quite simply and has the form

$$\varphi^{(0)}(t) = (\varepsilon + \alpha\lambda t)/(\lambda l) \tag{5.3}$$

The quantity l is given by formulae (3.1).

To construct solutions of the first two integral equations of (4.2) in analytical form, we will approximate the function $L(u)$ of the form (2.6) (or of the form (2.8)) by the expression

$$L^*(u) = u\sqrt{u^2 + C^2}/(u^2 + D^2) \tag{5.4}$$

where we choose the constants C and D in such a way as to satisfy the relations

$$L^*(u) = 1 + O(u^{-2}), \quad u \to \infty; \quad L^*(u) = lu + O(u^5), \quad u \to 0 \tag{5.5}$$

(compare with (3.1)).

Taking approximation (5.4) and expression (5.1) for the function $g(x)$ into account, the solutions of the first two integral equations of (4.2) can be constructed by the Wiener–Hopf method and have the form

$$\begin{aligned} \varphi^{(k)}(t) &= \frac{1}{\lambda}[\varepsilon - (-1)^{k+1}\alpha]\psi^{(0)}(t) + (-1)^{k+1}\alpha\psi^{(1)}(t), \quad k = 1, 2 \\ \psi^{(0)}(t) &= \frac{1}{l}\operatorname{erf}(\sqrt{Ct}) + \frac{\exp(-Ct)}{\sqrt{\pi l t}} \\ \psi^{(1)}(t) &= \frac{t}{l}\operatorname{erf}(\sqrt{Ct}) - \frac{\exp(-Ct)}{\sqrt{\pi C t}}\left(1 - \frac{D}{2C} - \frac{t}{l}\right) \end{aligned} \tag{5.6}$$

From formulae (3.7) and (3.10) we now obtain the quantities N_0 and N_1

$$\begin{aligned} N_0 &= 2\varepsilon\int_0^{2/\lambda}\psi^{(0)}(\tau)d\tau - \frac{2\varepsilon}{\lambda l} \\ N_1 &= -2\alpha\int_0^{2/\lambda}\psi^{(0)}(\tau)(\lambda\tau - 1)d\tau + 2\alpha\lambda\int_0^{2/\lambda}\psi^{(1)}(\tau)(\lambda\tau - 1)d\tau - \frac{2\alpha}{3\lambda l} \end{aligned} \tag{5.7}$$

Table 1

A	$\lambda=8$	$\lambda=4$	$\lambda=2$		$\lambda=1$	$\lambda=1/2$
N_0/ε						
2	1.429	2.033	3.157	3.338	5.747	10.63
1	1.293	1.786	2.738	2.899	4.859	8.832
1/2	1.207	1.633	2.477	2.636	4.332	7.764
N_1/ε						
2	1.594	1.659	1.897	1.981	2.671	4.196
1	1.584	1.621	1.763	1.866	2.409	3.623
1/2	1.578	1.601	1.692	1.800	2.257	3.288

Substituting expressions (5.6) for the functions $\psi^{(0)}(t)$ and $\psi^{(1)}(t)$ into relations (5.7) and evaluating the integrals, we obtain

$$\frac{N_0}{\varepsilon} = \frac{2}{\lambda l}[2\operatorname{erf}(\Lambda)-1]+\frac{1}{\sqrt{Cl}}\left(2-\frac{1}{\sqrt{Cl}}\right)\operatorname{erf}(\Lambda)+\frac{2}{l}\sqrt{\frac{2}{\pi C\lambda}}\exp(-\Lambda^2)$$

$$\begin{aligned}\frac{N_1}{\alpha} &= \frac{2}{3\lambda l}[2\operatorname{erf}(\Lambda)-1]+\left[\frac{1}{\sqrt{Cl}}\left(1-\frac{1}{\sqrt{Cl}}\right)+\right.\\ &+\left.\frac{1}{C}\left(2-\frac{D}{C}-\frac{1}{\sqrt{Cl}}+\frac{1}{2Cl}\right)\lambda+\frac{1}{C^2}\left(-1+\frac{D}{2C}+\frac{1}{4Cl}\right)\lambda^2\right]\operatorname{erf}(\Lambda)+\\ &+\left[\frac{2\sqrt{2}}{3l\sqrt{\pi Cl}}+\frac{\sqrt{2}}{C\sqrt{\pi l}}\left(2-\frac{5}{3\sqrt{Cl}}\right)\sqrt{\lambda}+\right.\\ &+\left.\frac{\sqrt{2}}{C\sqrt{\pi C}}\left(2-\frac{D}{C}-\frac{1}{2Cl}\right)\lambda^{3/2}\right]\exp(-\Lambda^2);\quad \Lambda=\sqrt{\frac{2C}{\lambda}}\end{aligned} \tag{5.8}$$

We recall that the quantities d_0, d_1, d_2 and d_3 in formulae (5.2) and the quantities, C, D and l in formulae (5.8) depend on the mechanical constants A and B of the form (2.2). We will present the results of a calculation for $B=1$ for three versions

$$\begin{aligned}&1)\ A=2\ (A^2>B);\quad C=1.2247,\quad D=1.7321,\quad l=0.40825\\ &d_0=0.58817,\quad d_1=-0.94427,\quad d_2=0.54479,\quad d_3=-0.36100\\ &2)\ A=1\ (A^2=B);\quad C=1,\quad D=\sqrt{2},\quad l=1/2\\ &d_0=0.35168,\quad d_1=-0.52104,\quad d_2=0.13495,\quad d_3=-0.034577\\ &3)\ A=1/2\ (A^2<B);\quad C=0.86603,\quad D=1.2247,\quad l=0.57735\\ &d_0=0.17369,\quad d_1=-0.30177,\quad d_2=0.017694,\quad d_3=0.0059917\end{aligned} \tag{5.9}$$

In Table 1 we give the values of the quantities N_0/ε and N_1/α, obtained from formulae (5.2) (they are shown on the left of the vertical dashed line) and from formulae (5.8) (shown on the right) for the above three versions. It can be seen that the asymptotic solutions for large and small λ begin to close in at approximately $\lambda=2$.

6. Formulation of the crack problem

Suppose an elastic orthotropic strip of thickness $2h$ occupies the region $|x|<\infty$, $-h\le y\le h$. The boundaries of the strip $y=-h$ and $y=h$ are hinged, while on the $y=0$ axis there is a crack of length $|x|\le a$. The crack sides are loaded with a pressure $q(x)$. We will henceforth confine ourselves to considering the case when $q(x)$ is an even function.

By virtue of what was said above, for the crack problem we will have Eqs. (1.2)–(1.5) and the following boundary conditions

$$\begin{aligned} &\upsilon(x, \pm h) = \tau_{xy}(x, \pm h) = 0, \quad \tau_{xy}(x, 0) = 0; \quad |x| < \infty \\ &\upsilon(x, 0) = 0, \quad |x| > a; \quad \sigma_y(x, 0) = -q(x), \quad |x| \le a \end{aligned} \tag{6.1}$$

7. The integral equation of the crack problem

We again introduce the Airy function by relations (2.1) and arrive at the differential Eq. (2.2).

Taking into account the symmetry of problem (6.1) about the y coordinate, we will consider initially the auxiliary problem for a strip of thickness h with boundary conditions

$$\begin{aligned} &\upsilon(x, h) = \tau_{xy}(x, h) = 0, \quad \tau_{xy}(x, 0) = 0, \quad \upsilon(x, 0) = \tilde{\gamma}(x); \quad |x| < \infty \\ &\tilde{\gamma}(x) = \gamma(x), \quad |x| \le a; \quad \tilde{\gamma}(x) = 0, \quad |x| > a \end{aligned} \tag{7.1}$$

Investigating the Airy function in the form of Fourier integral (2.4) and assuming for the present that $A^2 > B$, we obtain expression (2.5) for the Fourier transform $\Phi^*(y, \alpha)$ of the Airy function $\Phi(x, y)$. By then determining the function $c_k(\alpha)$ in this expression by satisfying boundary conditions (7.1), we finally obtain the following solutions of the auxiliary problem

$$\sigma_y(x, 0) = -\frac{\theta}{2\pi}\int_{-\infty}^{\infty}\Gamma(\alpha)\frac{\alpha}{L(\alpha h)}e^{-i\alpha x}d\alpha, \quad \Gamma(\alpha) = \int_{-a}^{a}\gamma(\xi)e^{i\alpha\xi}d\xi \tag{7.2}$$

where $\Gamma(\alpha)$ is the Fourier transform of the discontinuous function $\tilde{\gamma}(x)$ of the form (7.1), and the function $L(u)$ and the quantity θ are given by formulae (2.6).

We will now return to the main problem. Satisfying the last boundary condition of (6.1) using relations (7.2) (the remaining conditions of (6.1) have already been satisfied during the course of obtaining relation (7.2)), we arrive at an integral equation in the derivative of the function $\gamma(x)$, describing the displacements of points on the crack sides,

$$\int_{-a}^{a}\gamma'(\xi)M\left(\frac{\xi - x}{h}\right)d\xi = -\frac{\pi h}{\theta}q(x), \quad |x| \le a; \quad M(t) = \int_0^{\infty}\frac{1}{L(u)}\sin ut\,du \tag{7.3}$$

Expressions for the function $L(u)$ and the quantity θ for the case $A^2 = B$ and the case $A^2 < B$ have the form (2.8). Integral Eq. (7.3) does not change its form for these cases.

8. The asymptotic solution for a considerable relative thickness of the strip

On the basis of relations (3.1), the kernel of integral Eq. (7.3) can be represented in the form[2]

$$M(t) = \frac{1}{t} - \sum_{i=0}^{\infty}a_i t^{2i+1}, \quad a_i = \frac{(-1)^i}{(2i+1)!}\int_0^{\infty}\left[1 - \frac{1}{L(u)}\right]u^{2i+1}du \tag{8.1}$$

The series in expression (8.1) for $M(t)$ is absolutely convergent when $|t| < 2\kappa$.

Changing in integral Eq. (7.3) to the dimensionless variables

$$\xi' = \frac{\xi}{a}, \quad x' = \frac{x}{a}, \quad \varphi(x') = \gamma'(x), \quad p(x') = \frac{q(x)}{\theta}, \quad \lambda = \frac{h}{a} \tag{8.2}$$

and substituting expression (8.1) for $M(t)$ into it, we will have

$$\int_{-1}^{1}\frac{\varphi(\xi)}{\xi - x}d\xi = -\pi p(x) + \sum_{i=0}^{\infty}\frac{a_i}{\lambda^{2i+2}}\int_{-1}^{1}\varphi(\xi)(\xi - x)^{2i+1}d\xi \tag{8.3}$$

here and henceforth the primes on the dimensionless variables will be omitted.

We will seek a solution of integral Eq. (8.3) in the form of a series (3.5) by substituting expression (3.5) into integral Eq. (8.3) in the form of series (3.5). Substituting expression (3.5) into integral Eq. (8.3) and equating like powers of λ^{-2} on the left and right, we again arrive[2] at an infinite system of successively solvable integral equations in the functions $\varphi_n(x)$. We present the result of this solution, taking into account the evenness of the function $p(x)$

$$\begin{aligned}\varphi(x) &= \frac{1}{\pi\sqrt{1-x^2}}\int_{-1}^{1}\sqrt{1-t^2}p(t)\Big(\frac{1}{t-x}-\frac{a_0x}{\lambda^2}-\\ &-\frac{1}{2\lambda^4}[a_0^2x+2a_1(-x+xt^2+3x^3)]-\frac{1}{4\lambda^6}\{a_0^3x+a_0a_1x(1+2t^2+6x^2)+\\ &+a_2[-(11+18t^2-4t^4)x+(10+40t^2)x^3+20x^5]\}\Big)dt+O\Big(\frac{1}{\lambda^8}\Big)\end{aligned} \tag{8.4}$$

9. The asymptotic solution for a small relative thickness of the strip

We will confine ourselves to constructing the principal term of the asymptotic form of the solution of integral Eq. (7.3). It is given by expression (4.1), where, in view of the fact that the function $p(x)$ is even, we have

$$\varphi^{(2)}(t) = -\varphi^{(1)}(t) \tag{9.1}$$

while the functions $\varphi^{(1)}(t)$ and $\varphi^{(0)}(t)$ must now be found from the integral equations

$$\begin{aligned}&\int_0^\infty\varphi^{(1)}(\tau)M(\tau-t)d\tau = -\pi p(\lambda t-1), \quad 0\le t<\infty\\ &\int_{-\infty}^{\infty}\varphi^{(0)}(\tau)M(\tau-t)d\tau = -\pi p(\lambda t), \quad -\infty<t<\infty\end{aligned} \tag{9.2}$$

The solution of the first integral equation of (9.2) can be found by the Wiener–Hopf method[4]. The solution of the second integral equation of (9.2) can be found using the convolution theorem for a Fourier integral transformation.[5]

10. Numerical example

We will confine ourselves to considering the case when a uniform pressure acts on the crack sides, i.e. when

$$p(x)\equiv p = \text{const}, \quad p = q/\theta \tag{10.1}$$

Substituting expression (10.1) into formula (8.4), we obtain

$$\begin{aligned}\varphi(x) &= -\frac{px}{\sqrt{1-x^2}}\Big\{1+\frac{a_0}{2\lambda^2}+\frac{1}{4\lambda^4}\Big[a_0^2+6a_1\Big(-\frac{1}{4}+x^2\Big)\Big]+\\ &+\frac{1}{8\lambda^6}\Big[a_0^3+3a_0a_1\Big(\frac{1}{2}+2x^2\Big)+20a_2\Big(-\frac{3}{4}+x^2+x^4\Big)\Big]\Big\}+O\Big(\frac{1}{\lambda^8}\Big)\end{aligned} \tag{10.2}$$

The solution of the second integral equation of (9.2) for case (10.1) is equal to zero.

To construct a solution of the first integral equation of (9.2) in analytical form, we will again use an approximation of the function $L(u)$ of the form (5.4), (5.5). Taking this approximation and expression (10.1) for the function $p(x)$ into account, the solution of the first integral equation of (9.2) can be constructed by the Wiener–Hopf method and has the form

$$\varphi^{(1)}(t) = p\sqrt{l}\Big\{\frac{\exp(-Ct)}{\sqrt{\pi t}}-\frac{2}{\sqrt{\pi}}\sqrt{D-C}\exp(-Dt)\mathrm{F}(\sqrt{(D-C)t})\Big\} \tag{10.3}$$

Table 2

A	$\lambda = 8$	$\lambda = 4$	$\lambda = 2$		$\lambda = 1$	$\lambda = 1/2$
$N/(q\sqrt{a})$						
2	0.686	0.631	–	0.510	0.360	0.255
1	0.694	0.658	0.544	0.564	0.399	0.282
1/2	0.698	0.673	0.591	0.606	0.429	0.303

$$\mathrm{F}(x) = \int_0^x \exp(u^2)du$$

Bearing in mind the force fracture criterion,[6] we determine the normal stress intensity factor at the crack tip (along its extension) from the formula[2]

$$N = -\lim(\theta\gamma'(x)\sqrt{a-x}) \quad \text{as} \quad x \to a \tag{10.4}$$

On the basis of expression (10.2) we will have

$$\begin{aligned} \frac{N}{q\sqrt{a}} &= \frac{1}{\sqrt{2}}\left[1 + \frac{a_0}{2\lambda^2} + \frac{1}{4\lambda^4}\left(a_0^2 + \frac{9}{2}a_1\right) + \right. \\ &\left. + \frac{1}{8\lambda^6}\left(a_0^3 + \frac{15}{2}a_0a_1 + 25a_2\right)\right] + O\left(\frac{1}{\lambda^8}\right) \end{aligned} \tag{10.5}$$

while on the basis of expressions (4.1) and (10.3) we obtain

$$\frac{N}{q\sqrt{a}} = \sqrt{\frac{\lambda l}{\pi}} \tag{10.6}$$

The quantities a_0, a_1 and a_2 in formula (10.5) and l in formula (10.6) depend on the mechanical constants A and B of the form (2.2). For l we take the values shown in formulae (5.9). For the values of a_0, a_1 and a_2 for the versions of the relation between the constant A and B indicated in Section 5 for $B = 1$, we obtain

$$\begin{aligned} &1)\ A = 2\ (A^2 > B); \quad a_0 = -4.1123, \quad a_1 = 2.5705, \quad a_2 = -2.2572 \\ &2)\ A = 1\ (A^2 = B); \quad a_0 = -2.4674, \quad a_1 = 0.67645, \quad a_2 = -0.22254 \\ &3)\ A = 1/2\ (A^2 < B); \quad a_0 = -1.6449, \quad a_1 = 0.13529, \quad a_2 = 0.031792 \end{aligned} \tag{10.7}$$

In Table 2 we give values of the quantity $N/(q\sqrt{a})$, obtained from formula (10.5) (they are shown on the left of the vertical dashed line) and from formula (10.6) (shown on the right) for the above three versions. It can be seen that the asymptotic solutions begin to approach one another for large and small values of λ, as previously, approximately when $\lambda = 2$. For version $A = 2$, the closure begins when $\lambda = 2.8$ with an error of 6.6% (this is expressed by the absence of terms of the order of λ^{-8} in (10.5)).

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